be of interest to assembly line managers who are contemplating a structural change from a completely synchronous design to a mixed-transfer mode configuration.

REFERENCES


An Extension to Operational Space for Kinematically Redundant Manipulators: Kinematics and Dynamics

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Abstract—An extension to operational space (EXOS) is presented for the explicit representation of the null-space (NS) dynamics and its interaction with the operational-space dynamics. First, the EXOS Jacobian is formed by augmenting the Jacobian matrix with a minimum number of its NS vectors. Based on the EXOS Jacobian, free of algorithmic singularity, the kinematics, statics, and dynamics of a redundant manipulator are derived in a compact form. In particular, the resulting EXOS dynamics is able to identify the inner dynamic structure. Its efficacy and efficiency have been demonstrated through comparative analysis and simulation.

Index Terms—Dynamics, kinematics, null space, operational space, redundant manipulator, statics.

I. INTRODUCTION

The operational-space (OS) formulation [1] provides a comprehensive framework to describe the end-effector dynamic behavior. More specifically, it enables a better understanding and control of the end-effector motion—the respective motion in each direction in the OS—as well as its interaction (coupling) with other motions.

II. EXTENDED OPERATIONAL SPACE

The kinematic equation for a redundant manipulator is given as

$$\dot{x} = J\dot{\theta}$$  \hspace{1cm} (1)$$

where $x \in \mathbb{R}^m$ denotes the location of end effector with respect to the base frame, $\theta \in \mathbb{R}^n$ joint vector, and $J$ a vector consisting of $n$ scalar functions. Here, $n > m$ and the degree of redundancy ($n - m$) is denoted by $r$. For dimensional consistency, it is assumed without loss of generality that all joints are revolute.\footnote{This paper and [4] are based on almost verbatim translations of [2] and [3]. The difference, however, is that: in [4] we added experimental results; in this paper, a substantial enhancement in analysis.}

A. EXOS Concept

From the kinematic equation, the Jacobian equation of the manipulator is determined as

$$\dot{x} = J\dot{\theta}$$  \hspace{1cm} (2)$$

where $(\dot{})$ denotes the time derivative, and $J = (\partial f) / (\partial \theta) \in \mathbb{R}^{m \times n}$ denotes a Jacobian matrix. Now, define $Z \in \mathbb{R}^{r \times n}$ as a matrix consisting of the orthonormal basis vectors spanning the NS (or the tangent space of the self-motion manifold). Then, $Z$ satisfies the following relationship:

$$ZZ^\top = I$$  \hspace{1cm} (3)$$

$$ZZ^\top J = 0$$  \hspace{1cm} (4)$$

In addition, define $x_N$ as the NS velocity. Then, we have a complementary mapping relationship at velocity level between the joint space and the NS, which is

$$x_N \equiv Z\dot{\theta}.$$  \hspace{1cm} (5)$$

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By using (2) and (5), let us define the EXOS as a space that consists of \( m \)-dimensional OS for end-effector motion and \( r \)-dimensional NS for self-motion. Then, EXOS Jacobian can be defined as

\[
J_E \equiv \begin{bmatrix} J \\ Z \end{bmatrix}
\]  

(6)

and EXOS velocity as

\[
\dot{x}_E \equiv \begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix}.
\]

(7)

Now, the Jacobian equation is determined as

\[
\dot{x}_E = J_E \dot{\theta}.
\]

(8)

Note that the EXOS Jacobian equation enables to treat the differential kinematics of redundant manipulators as if they were nonredundant manipulators.

The relationship (8) can be described more clearly by the visualization Fig. 1. The direction, the component responsible for self-motion. The remaining part of \( \dot{\theta} \) corresponds to the component contributing to the motion in the OS.

Note that \( Z \) can be obtained by using either the singular value decomposition method [11] or the method in [10].

B. Important Properties of EXOS Jacobian, \( J_E \)

By using the definition of EXOS Jacobian, \( J_E \), in (8) and \( Z \) in (4), we can derive the determinant of \( J_E \) as follows. Since

\[
J_E J_E^T = \begin{bmatrix} J^T \\ Z^T \end{bmatrix} \begin{bmatrix} J \end{bmatrix}^T = \begin{bmatrix} J J^T & 0 \\ Z Z^T & 0 \end{bmatrix}
\]

(9)

it immediately follows that

\[
\det(J_E J_E^T) = \det(J J^T) \det(Z Z^T).
\]

(10)

From (10) and (4), follows the determinant of EXOS Jacobian

\[
\det(J_E) = \sqrt{\det(J J^T)}.
\]

(11)

Equation (11) shows that the EXOS Jacobian has the following properties.

- \( J_E \) is singular only when \( \det(J J^T) = 0 \), which occurs if and only if the manipulator is kinematically singular. The EXOS Jacobian behaves at singularity in the same way as a Jacobian matrix does for nonredundant manipulators.
- The determinant is the same as the manipulability measure [13].
- At any configuration without kinematic singular point, there is no other singularity. Hence, no algorithmic singularity exists in EXOS Jacobian.

III. EXOS KINEMATICS

The forward kinematics is available from the defining relationships for EXOS as

\[
\begin{bmatrix} \dot{x}_d \\ \dot{x}_{N,d} \end{bmatrix} = J_E \dot{\theta}_i
\]

(12)

where \( \dot{x}_d, \dot{x}_{N,d}, \) and \( \dot{\theta}_i \) are the desired end-effector velocity, the NS velocity, and the corresponding joint velocity, respectively. The inverse kinematics, then, is expressed as

\[
\dot{\theta}_i = J_E^{-1} \begin{bmatrix} \dot{x}_d \\ \dot{x}_{N,d} \end{bmatrix}.
\]

(13)

If a secondary performance is required and specified with a measure of \( H \), then the desired NS velocity may be defined as

\[
\dot{x}_{N,d} = Z \dot{h}
\]

(14)

where

\[
h = k_h \nabla H
\]

(15)

with \( k_h \) being a constant. Combining this equation with (13) leads to

\[
\dot{\theta}_i = J_E^{-1} \begin{bmatrix} \dot{x}_d \\ Z \dot{h} \end{bmatrix}.
\]

(16)

A. Relationship with Resolved Motion Method

Resolved motion method [14] (RMM) is given as

\[
\dot{\theta}_i = J^+ \dot{x}_d + (I_n - J^+ J) \dot{h}
\]

(17)

where \( J^+ = J^T (J J^T)^{-1} \) is Moore–Penrose pseudo-inverse matrix of \( J \).

In [15], the following has been derived:

\[
J^+ = J_E^{-1} \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad I_n - J^+ J = J_E^{-1} \begin{bmatrix} 0 \\ Z \end{bmatrix}.
\]

(18)

Combining the above two equations with (17) leads to (16) [15].

Hence, (17) and (16) are essentially the same, sharing the same inverse kinematic solutions. Yet, in addition to having more compact form, using (16) may have some significant advantages stemming from its very expression as follows.

- In RMM, since the NS projection matrix, the rank of which is \( r \), is an \( n \times n \) matrix, there are \( m \) overlapping equations. In comparison, since the NS basis matrix in EXOS has exactly \( r \) vectors, the method tends to expose the contribution of each vector rather transparently. This advantage will be clearly demonstrated in Section IV, where EXOS formulation yields the dynamics equations in the more succinct form.
- With EXOS method, we can treat kinematics, statics, dynamics, and control in a consistent manner as if the manipulator were nonredundant, thereby sharing insights already established, such as the duality of kinematics and statics.

B. Comparison with Extended Jacobian Method

As is well known, the extended Jacobian method (EJM) [16] is given as follows:

\[
\dot{\theta}_i = \left[ \frac{\partial J}{\partial (Z \nabla H)} \right]^{-1} \begin{bmatrix} \dot{x}_d \\ 0 \end{bmatrix}.
\]

(19)

It is easy to observe that (16) and (19) have similarity in their forms, yet difference in their augmented terms. Note, however, that \( \frac{\partial (Z \nabla H)}{\partial \theta} \) does not represent the NS of \( J \). Since our objective is to obtain a framework for both the OS dynamics and NS dynamics, the formulation due to EJM does not serve to this objective, and hence is not considered any more.
IV. EXOS DYNAMICS

In this section, we will derive the statics and dynamics by using the framework of EXOS.

A. EXOS Statics

In order for a redundant manipulator to stay in a perfect equilibrium state, the virtual displacement needs to exist not only in the OS but also in the NS. Let $\mathcal{F}_N$ denote the null force responsible for the NS displacement $\delta x_N$.

Virtual work for the redundant manipulator is expressed as

$$\delta W = \tau \cdot \delta \theta - \mathcal{F} \cdot \delta x - \mathcal{F}_N \cdot \delta x_N = 0$$

where $(\cdot)$ denotes the inner product of vector. Alternatively

$$\tau^T \delta \theta = \mathcal{F}^T \delta x + \mathcal{F}_N^T \delta x_N.$$  \hfill (21)

Since $\delta x_E = J_{E} \delta \theta$ from (8), combining this equation with (21) and then transposing, we have

$$\tau = J_{E}^T \mathcal{F}_E.$$  \hfill (22)

where

$$\mathcal{F}_E = \left\{ \mathcal{F}, \mathcal{F}_N \right\}.$$  \hfill (23)

Note that (8) and (22) show the duality between the kinematics and statics in the same manner as in the nonredundant case.

B. EXOS Dynamics

To provide the context for the EXOS dynamics, let us first present the joint space dynamics and the OS dynamics, respectively

$$\tau = J_{\theta} \dot{\theta} + N_\theta \dot{\theta}$$

$$\mathcal{F} = M_{\theta} \ddot{x} + N_x$$

where $M_\theta \in \mathbb{R}^{n \times n}$ denotes an inertia matrix, and $N_\theta \in \mathbb{R}^{n}$ denotes a vector which includes centrifugal, coriolis, gravitational, and frictional force, etc.; $M_x = J_{E}^T M_\theta J_{E}^{-1} (\in \mathbb{R}^{m \times m})$ and $N_x = J_{E}^T (N_\theta - M_\theta J_{E}^{-1} J_{E} \dot{\theta}) (\in \mathbb{R}^{m})$, with $J_{E}$ denoting the inertia weighted pseudo-inverse [1] defined as

$$J_{E}^{-1} = M_\theta^{-1} J_{E}^T (M_\theta^{-1} J_{E}^T)^{-1}.$$  \hfill (26)

To derive the EXOS dynamics, differentiate the Jacobian equation (8) and solve for joint acceleration as follows:

$$\dot{\theta} = J_{E}^{-1} (\ddot{x}_E - J_{E} \dot{\theta}).$$  \hfill (27)

Then, from the statics in (22) and joint dynamics in (24), we have

$$\mathcal{F}_E = J_{E}^{-T} (M_\theta \ddot{x} + N_\theta).$$  \hfill (28)

Substituting (27) into (28) leads to the following EXOS dynamics:

$$\mathcal{F}_E = M_{\theta E} \ddot{x}_E + N_{x_E}$$

where $M_{\theta E} = J_{E}^{-T} M_\theta J_{E}^{-1} (\in \mathbb{R}^{n \times n})$, $N_{x_E} = J_{E}^{-T} (N_\theta - M_\theta J_{E}^{-1} J_{E} \dot{\theta}) (\in \mathbb{R}^{n})$.

C. Insights from EXOS Dynamics

Now, let us express the EXOS dynamics as the following:

$$\ddot{x}_E = M_{\theta E}^{-1} (\mathcal{F}_E - N_{x_E})$$  \hfill (30)

where the inverse of EXOS inertia matrix is determined as

$$M_{\theta E}^{-1} = \begin{bmatrix} J_{E}^{-1} \mathcal{F} & J_{E}^{-1} \mathcal{F}_N \\ Z_{M_\theta^{-1}} J_{E}^T & Z_{M_\theta^{-1}} J_{E}^T \end{bmatrix}.$$  \hfill (31)

Evidently this matrix maps a set of OS force and NS force into a corresponding set of OS motion and NS motion in various combinations as follows.

- $J_{E}^{-1} \mathcal{F}$ determines (or maps into) OS motion from OS force describing the OS dynamics only—one can confirm in (25).
- $Z_{M_\theta^{-1}} J_{E}^T$ determines (or maps into) NS motion from NS force, hence describing self-motion as such.
- $J_{E}^{-1} \mathcal{F}_N$ determines (or maps into) OS motion from NS force, representing an interaction (or coupling) between OS dynamics and NS dynamics.
- $Z_{M_\theta^{-1}} J_{E}^T$ determines (or maps into) NS motion from OS force, representing another interaction (or coupling) between OS dynamics and NS dynamics.

In short, EXOS dynamics exposes the inner structure: the two block matrices on the diagonal represent the OS and NS dynamics, respectively; the two blocks on the off-diagonal represent the interactions between the two dynamics. This point will be illustrated by a simulation in Section VI.

If we use $Z_{M_\theta}$ in (31) instead of $Z$, $M_{E}^{-1}$ becomes

$$M_{E}^{-1} = \begin{bmatrix} J_{E}^{-1} \mathcal{F} & 0 \\ 0 & Z_{M_\theta} J_{E}^T \end{bmatrix}$$  \hfill (32)

which explains why OS dynamics is decoupled from NS dynamics when the inertia weighted pseudo-inverse is used for redundancy resolution [1].

V. COMPARISON WITH OTHER METHODS

Having proposed the EXOS dynamics and expounded its significance, it is instructive to compare it with other formulations in terms of the manner and the economy with which the OS dynamics and NS dynamics are described. The methods of interest are the dynamics due to the RMM and the OS formulation, respectively. To this end, we have derived each of the dynamics in a form similar to the EXOS dynamics in (31).

A. Dynamics Due to the RMM

Since the NS matrix $J_{NS}^{-1}$ for the RMM is given as $J_{NS}^{-1} = (J_{n} - J_{\theta} J_{\theta}^T)$, the NS acceleration $\dot{\phi}_{\theta}$ satisfies $\dot{\phi}_{\theta} = J_{\theta} \dot{\theta}$. In addition, it holds that $\tau = J_{\theta}^T \mathcal{F} + J_{\theta}^T \Gamma_{r_{m}}$, with $\Gamma_{r_{m}}$ denoting the null force. Making the NS terms explicit as

$$\begin{bmatrix} \dot{x}_{r_{m}} \\
\dot{\phi}_{\theta} \\
\dot{\Gamma}_{r_{m}} \\
\dot{\mathcal{F}} \\
\dot{J}_{\theta} \\
\dot{J}_{NS} \end{bmatrix} = \begin{bmatrix} \mathcal{F} \\
\Gamma_{r_{m}} \\
\Gamma_{r_{m}}^T \\
\mathcal{F}_{\theta}^T \\
J_{\theta}^T \\
J_{\theta}^T \end{bmatrix}.$$  \hfill (33)

we obtain the inverse inertia matrix as follows:

$$M_{\theta E}^{-1} = J_{r_{m}}^T M_{\theta E}^{-1} J_{r_{m}}^{-1} = \begin{bmatrix} J_{E}^{-1} \mathcal{F} & J_{E}^{-1} \mathcal{F}_N \\ Z_{M_\theta^{-1}} J_{E}^T & Z_{M_\theta^{-1}} J_{E}^T \end{bmatrix}.$$  \hfill (34)

Through (34), we could explicitly include both the OS part and NS part, as well as their couplings. Note that the NS part in (34) has a dimension of $n \times n$ instead of $r \times r$, implying that more relationships are provided than necessary. As the result, the inertia matrix has the dimension of $(n + m) \times (n + m)$ instead of $n \times n$. For example, a 3-DOF planar manipulator with $m = 2 \ (r = 1)$ has only one-dimensional NS. Yet
the inertia matrix in (34) has a dimension of $5 \times 5$, instead of $3 \times 3$ in the EXOS case; its NS part consists of nine elements, making it difficult to understand the NS dynamics.

A close inspection of derivation procedure reveals that this problem is inevitable as long as $(I_n - J^- J)$ is used as the NS matrix. This shows the importance of starting with a compact NS matrix such as $Z$ at kinematic level and thus the usefulness of EXOS approach.

### B. Dynamics Due to the OS Formulation

From the statics of the OS formulation [1], $\tau = J^T \mathcal{F} + (I_n - J^- J) \ddot{\theta}$, it is clear that $(I_n - J^- J) \in \mathbb{R}^{n \times n}$ is used as its NS matrix. Let $J_{\text{NS}} = (I_n - J^- J)$, and let $\phi$ denote the acceleration in the NS, which then satisfies $\dot{\phi} = J_{\text{NS}}^{-1} \ddot{\theta}$.

Include explicitly the NS part force and acceleration as follows:

$$\ddot{x}_{\phi} = \begin{bmatrix} \ddot{x} \\ \dot{\phi} \end{bmatrix}, \quad \mathcal{F}_{\phi} = \begin{bmatrix} \mathcal{F} \\ \Gamma \end{bmatrix}, \quad J_{\phi} = \begin{bmatrix} J \\ J_{\text{NS}}^{-1} \end{bmatrix}. \quad (35)$$

Then, it is easy to obtain the inverse inertia matrix

$$M_{\phi}^{-1} = J_{\phi} M_{\phi}^{-1} J_{\phi}^T = \begin{bmatrix} J M_{\phi}^{-1} J^T & J M_{\phi}^{-1} J_{\text{NS}}^{-1} \\ J_{\text{NS}}^{-1} M_{\phi}^{-1} J & J_{\text{NS}}^{-1} M_{\phi}^{-1} J_{\text{NS}}^{-1} \end{bmatrix}.$$ \quad (36)

$$= \begin{bmatrix} J M_{\phi}^{-1} J^T & 0 \\ 0 & J_{\text{NS}}^{-1} M_{\phi}^{-1} J_{\text{NS}}^{-1} \end{bmatrix}. \quad (37)$$

As with the RMM, the OS dynamics in (25) is now made to explicitly include both the OS part and NS part. Yet, the NS part again has a dimension of $n \times n$ instead of $r \times r$, sharing the same problem with the RMM case. Like the RMM case, this problem comes directly from the way the NS is constructed. If $ZM_0$ were used instead of $(I_n - J^- J)$, (32) would result.

### VI. Simulation

To illustrate the point in (31), let us consider a planar 3-DOF redundant manipulator in Fig. 2. Each link has the length of 1 m, the mass of 12 kg, and the inertia of 1 kgm². Fig. 3 shows the elements of $M_{\phi}^{-1}$ at each joint configuration, $\alpha = \theta_1 + \theta_2 + \theta_3$, when the end effector stays at $(x, y) = (2, 0)$ with the joint changing its configuration along the self-motion manifold ($r = 1$). In this figure, $\rho_{ij}$ denotes the element of $i$th row and $j$th column of $M_{\phi}^{-1}$. More specifically, Fig. 3(a) on one hand shows the OS force-acceleration relationships: $\rho_{11}$ denote the relationship in $x$ direction, $\rho_{22}$ that in $y$ direction, and $\rho_{12}$ stands for the dynamic coupling between $x$-axis and $y$-axis. Fig. 3(b) on the other hand shows the elements of $M_{\phi}^{-1}$ owing to the self-motion. That is, $\rho_{33}$ stands for the NS force-acceleration relationship, and $\rho_{13}, \rho_{23}$ the dynamic coupling between OS and NS. Note that the inner dynamic structure has been made transparent through the EXOS framework.

The simulation above can be also used to illustrate the comparison in Section V. Fig. 4(a) shows the six elements representing the NS part of $M_{\phi}^{-1}$ in (34) and Fig. 4(b) those of $M_{\phi}^{-1}$ in (36). In comparison, Fig. 4(c) shows $\rho_{33}$ of Fig. 3(b) representing the one-dimensional NS. Clearly, the EXOS formulation makes NS dynamics easier to interpret than the other two. Using $ZM_0$ instead of $(I_n - J^- J)$, we can change the six elements of Fig. 4(b) into one element as plotted in Fig. 4(d).

### VII. CONCLUSION

In this study, we have proposed a comprehensive framework, EXOS, which consists of OS vectors and the minimum number of NS vectors. In this framework, we formulated kinematics, statics, and dynamics of redundant manipulators in a consistent way. Including only the minimum number of NS vectors from the very beginning has rendered the resulting dynamic equations into a compact and succinct form.

As the results, the dynamic equations effectively and efficiently describe not only the dynamic behavior of the end effector, but also that of the self-motion; at the same time, the interaction between these two motions. The effectiveness was highlighted in the revelation of the dynamic couplings between OS dynamics and NS dynamics; the efficiency was confirmed through comparison to other methods transformed in similar forms.
Fig. 4. Elements of inverse of inertia matrix along the self-motion manifold for showing NS dynamics of: (a) RMRC, (b) OS, (c) EXOS, and (d) EXOS with $ZM_0 [\theta_1 + \theta_2 + \theta_3]$ with the end-effector fixed at $(x, y) = (2, 0)$.

REFERENCES